

Do we have a general eigenvalue condition in Quantum Mechanics ? : No

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Abstract

Here we clearly notice that invariance in commutation relation can be associated with complex energy eigenvalues .Hence present work clearly demonstrates that as such we,do not have a general eigenvalue condition in Quantum Physics.Here it is shown that existing all the three eigenvalue conditions namely: Hermiticity , PT symmetry and Pseudo-Hermiticity ,are not sufficient enough even to explain the spectra of Harmonic Oscillator under non-Hermitian transformation of momentum,co-ordinate or both, on appearance of complex energy eigenvalues .In fact complex energy eigenvalues become an inherent feature of transformed oscillator, when its frequency of oscillation matches with that of variational based calculation.

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I.Introduction

In the development of Quantum Mechanics , Hermiticity condition on Hamiltonians was proposed [1] to study of real spectra .However , Bender and Boettecher(BB) [2] proposed a simpler space-time invariance condition i.e PT symmetry in eigenvalue relation, in place of Hermiticity condition.In space-time relation we have : parity operation ($P \Rightarrow x \rightarrow -x$) and time reversal operation ($T \Rightarrow i \rightarrow -i$) . In a very recent calculation , Rath ,Mallick and Samal [3] pointed out that unbroken PT symmetry condition of BB[2] breaks down on a simple non-Hermitian transformation of co-ordinate i.e $x \rightarrow x - i\lambda$ in the interaction of the type ix^3 . Hence we believe that neither the Hermiticity or PT symmetry does not satisfy the requirement for a general eigenvalue condition. Further a micro level observation on PT symmetry work[2] reveals that terms like $f(x)f(p)$ have not been considered in proclaiming the PT symmetry condition in place of Hermiticity condition. In fact realising this drawback, steps were taken to accomodate terms like [4]

$$f(x)f(p) + f(p)f(x) \tag{1}$$

in non-Hermitian Hamiltonian and propose a new condition . The new condition ,which bypasses the PT condition, is the Pseudo-Hermiticity condition proposed by Mostafazadeh [5] . The work of Mostafazadeh [5]] gained momentum particularly after the work Ahmed [6] ,who proposed that non-Hermitian transformation of momentum i.e

$$p \rightarrow p + iLx = y \tag{2}$$

in Harmonic Oscillator and showed the iso-spectral behaviour. One will notice that terms in Eq(1) are clearly observed in the work of Ahmed[6]i.e

$$i(xp + px) \tag{3}$$

which is not only PT invariance but also Pseudo-Hermiticity in nature . Further generalisation of non-Hermitian transformation was noticed in the work of Rath and

Mallick [7] in demonstrating the iso-spectral behaviour under simultaneous transformations of co-ordinate and momentum , using perturbation theory . Later on ,Fernandez [8] also extended the work of RM[7] using similarity transformation [8].Interestingly Rath [9], while studying terms as in Eq(1), noticed similar behaviour prposed in the work of Hatano-Nelson [10] . Let us highlight the visionary work of HN [10] as follows :

$$H = \frac{1}{2}[(p + ig)^2 + V(x)] \quad (4)$$

the energy eigenvalues must satisfy the conditions

$$E = E_{real}g < g_c \quad (5)$$

or

$$E = E_{imaginary}g > g_c \quad (6)$$

Now question arrises why the work of Ahmed [6] could not reflect the similar conclusions of Hatano-Nelson ?. However aim of this paper is to show that when the Hamiltonian contantains Eq(3) and the corresponing commutation relation remains invariant with the original one i.e

$$[x, p] = i \quad (7)$$

then ,the eigevalues of the system , can be complex in nature provided the frequency of oscillation matches with that calculated using variational principle (even if the transformed Hamiltonian is PT symmetry or Pseudo-Hermiticity in nature). In other words invariance in commutation relation can be associated with complex energy eigenvalues .

IIA.Non-Hermitian transformation and invariance of commutation relation. via MATLAB.

Let us make non-Hermitian transformation [7,8] of co-ordinate of x as

$$x \rightarrow \frac{x + iRp}{\sqrt{1 + LR}} \quad (8)$$

and momentum p as

$$p \rightarrow \frac{p + iLx}{\sqrt{1 + LR}} \quad (9)$$

in Harmonic Oscillator Hamiltonian . In the above choice of transformation we notice invariance in commutation relation i.e

$$[x, p] = \frac{1}{(1 + LR)}[x + iRp, p + iLx] = i \quad (10)$$

IIB MATLAB for commutation relation

Present a MATLABprogramme , which is a modified version of Korsch and Gluck[11] is now under test in getting correct result for terms like xp and px appearing in commutation relation.

MATLAB

$$N = 100; \quad (11)$$

$$s = 1; \quad (12)$$

$$n = 1 : N - 1; \quad (13)$$

$$m = \sqrt{(n)}; \quad (14)$$

$$L = 3; \quad (15)$$

$$R = 4; \quad (16)$$

$$x = \frac{s}{\sqrt{2}}(\text{diag}(m, -1) + \text{diag}(m, 1)); \quad (17)$$

$$p = \frac{is}{\sqrt{2}}(\text{diag}(m, -1) - \text{diag}(m, 1)); \quad (18)$$

$$C = \frac{1}{1 + LR} \quad (19)$$

$$y = (p + iLx); \quad (20)$$

$$z = (x + iRp); \quad (21)$$

$$H = \frac{C(z y - y z)}{i}; \quad (22)$$

$$EigSort = \text{sort}(\text{eig}(H)); \quad (23)$$

$$EigSort(1 : 50) \tag{24}$$

One will notice that for any value of L, R ,the commutation relation remains invariant.

IIIA. Non-Hermitian PT symmetrized and Pseudo-Hermitrized Hamiltonian .

Here we consider a Hamiltonian which is not only reflects PT symmetry but also Pseudo-Hermiticity condition under the influence of simultaneous transformation of momentum and co-ordinate as stated above i.e

$$H = \frac{1}{(1 + LR)} [A^2(p + iLx)^2 + B^2(x + iRp)^2] \quad (25)$$

This Hamiltonian is a general non-Hermitian Harmonic Oscillator . Usng suitable values of A, B ,one can generate any desired Hamiltonian which will satisfy not only PT symmetry [2] but also Pseudo-Hermiticity [5] condition . In its condensed one can write the Hamiltonian as

$$H = \frac{1}{(1 + LR)} [A^2y^2 + B^2z^2] \quad (26)$$

III. Frequency of Oscillation for Spectral instability: $w = w_v$ from variational method .

Now we use standard variational method [1,12] and determine the frequency of oscillation from diagonal term as follows :

$$\frac{d \langle n | H | n \rangle}{dw} = 0 \quad (27)$$

Here we get an interesting relation on $w = w_v$ where

$$w = w_v = \sqrt{\frac{B^2 - L^2 A^2}{A^2 - R^2 B^2}} \quad (28)$$

In the direct study approach we use different values of w and calculate the energy eigenvalues .

IV. Energy eigenvalues direct calculation using MATLAB .

In the present case we use the MATLAB programme ,which is a modified version of the earlier one as used above (in commutation relation) . study .

B – MATLAB

$$N = 100; \quad (29)$$

$$s = 1; \quad (30)$$

$$n = 1 : N - 1; \quad (31)$$

$$m = \sqrt{(n)}; \quad (32)$$

$$L = 3; \quad (33)$$

$$W = 4; \quad (34)$$

$$R = 0; \quad (35)$$

$$A = 1; \quad (36)$$

$$B = \sqrt{L^2 + W^2}; \quad (37)$$

$$w = W; \quad (38)$$

$$x = \frac{s}{\sqrt{(2w)}}(\text{diag}(m, -1) + \text{diag}(m, 1)); \quad (39)$$

$$p = \frac{is}{\sqrt{(2/w)}}(\text{diag}(m, -1) - \text{diag}(m, 1)); \quad (40)$$

$$C = \frac{1}{1 + LR} \quad (41)$$

$$y = (p + iLx); \quad (42)$$

$$z = (x + iRp); \quad (43)$$

$$H = C[A^2y^2 + B^2z^2]; \quad (44)$$

$$\text{EigSort} = \text{sort}(\text{eig}(H)); \quad (45)$$

$$\text{EigSort}(1 : 50) \quad (46)$$

One can change the value of w and study the eigenvalues .

V.Result and Discussion .

In table -I, we reflect instability in iso-spectral behaviour for $A = 1; B^2 = L^2 + W^2; R = 0$ same values of B, L, W using the frequency of oscillation $w = w_v = W$. The interesting point in this selection is that we reflect spectral instability in non-Hermitian Hamiltonian proposed by Ahmed [7] . In table -II, we reflect instability in iso-spectral behaviour for $B = 1; A^2 = R^2 + W^2; L = 0$ same values of R, W using the frequency of oscillation $w = w_v = \frac{1}{W}$. Similarly one can notice spectral instability behaviour for many Hamiltonians using suitable values of the parameters . In conclusion as such we do not have a general eigenvalue condition, which will be valid for all operators in Quantum Mechanics.

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Table -I :Instability in iso-spectra of H with $A = 1, R = 0$ and $B = \sqrt{W^2 + L^2}$.

W	L	$w = W$	$E_n \rightarrow H$	ϵ_n	Remarks
4	3	w	5	5	iso-spectra
4	3	w	15	15	iso-spectra
4	3	w	25	25	iso-spectra
4	3	w	35	35	iso-spectra
4	3	w	45	45	iso-spectra
4	3	w	55	55	iso-spectra
4	3	w	395.53-59.95i	445	No iso-spectra
4	3	w	395.53+59.95i	455	No iso-spectra
4	3	w	398.30-50.18i	465	No iso-spectra
4	3	w	398.30+50.18i	475	No iso-spectra
4	3	w	412.52-82.47i	485	No iso-spectra
4	3	w	412.52+82.47i	495	No iso-spectra
3	4	w	5	5	iso-spectra
3	4	w	15	15	iso-spectra
3	4	w	25	25	iso-spectra
3	4	w	35	35	iso-spectra
3	4	w	45	45	iso-spectra
3	4	w	55	55	iso-spectra
3	4	w	281.71-137.20 i	445	No iso-spectra
3	4	w	281.71+137.20 i	455	No iso-spectra
3	4	w	280.41-144.83 i	465	No iso-spectra
3	4	w	280.41+144.83 i	475	No iso-spectra
3	4	w	295.72-163.26 i	485	No iso-spectra
3	4	w	295.72+163.26 i	495	No iso-spectra

Table -II :Instability in iso-spectra of H with $B = 1, L = 0$ and $A = \sqrt{W^2 + R^2}$.

W	R	$w = \frac{1}{W}$	$E_n \rightarrow H$	ϵ_n	Remarks
4	3	w	5	5	iso-spectra
4	3	w	15	15	iso-spectra
4	3	w	25	25	iso-spectra
4	3	w	35	35	iso-spectra
4	3	w	45	45	iso-spectra
4	3	w	55	55	iso-spectra
4	3	w	395.53-59.95i	445	No iso-spectra
4	3	w	395.53+59.95i	455	No iso-spectra
4	3	w	398.30-50.18i	465	No iso-spectra
4	3	w	398.30+50.18i	475	No iso-spectra
4	3	w	412.52-82.47i	485	No iso-spectra
4	3	w	412.52+82.47i	495	No iso-spectra
3	4	w	5	5	iso-spectra
3	4	w	15	15	iso-spectra
3	4	w	25	25	iso-spectra
3	4	w	35	35	iso-spectra
3	4	w	45	45	iso-spectra
3	4	w	55	55	iso-spectra
3	4	w	281.71-137.20 i	445	No iso-spectra
3	4	w	281.71+137.20 i	455	No iso-spectra
3	4	w	280.41-144.83 i	465	No iso-spectra
3	4	w	280.41+144.83 i	475	No iso-spectra
3	4	w	295.72-163.26 i	485	No iso-spectra
3	4	w	295.72+163.26 i	495	No iso-spectra